Laboratory session 1

Implementation of random binning encoding and secrecy rate evaluation

Nicola Laurenti, Francesco Ardizzon November 20, 2020

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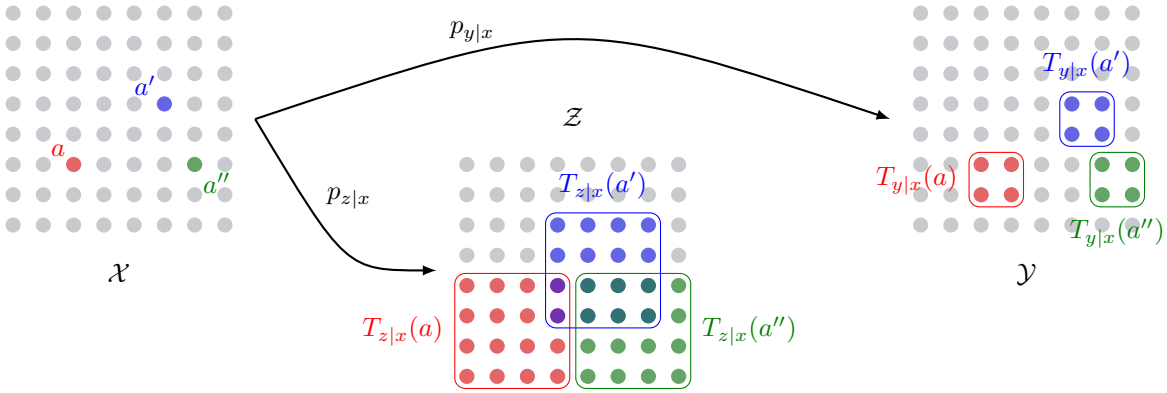
Laboratory session 1— Contents

Toy example: uniform channel

Consider a wiretap channel in which

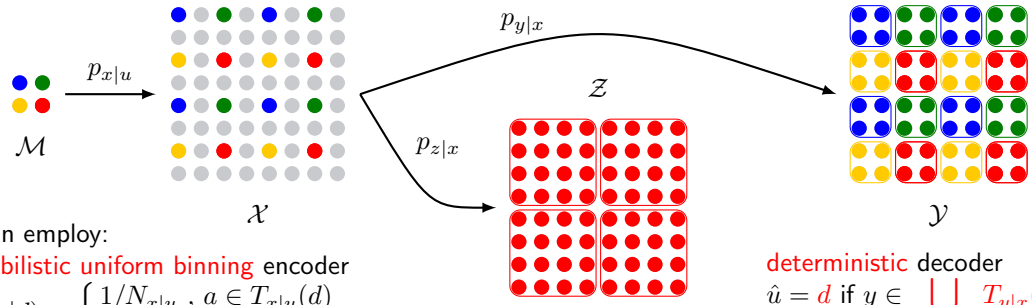
$$p_{y|x}(b|a) = \begin{cases} 1/N_{y|x} \ , \ b \in T_{y|x}(a) \\ 0 \qquad\quad , \ b \notin T_{y|x}(a) \end{cases} \quad , \quad p_{z|x}(c|a) = \begin{cases} 1/N_{z|x} \ , \ c \in T_{z|x}(a) \\ 0 \qquad\quad , \ c \notin T_{z|x}(a) \end{cases}$$



Random binning encoding

If we can find:

- a subset $\mathcal{X}' \subset \mathcal{X}$ such that $\forall a \neq a' \in \mathcal{X}'$, $T_{y|x}(a) \cap T_{y|x}(a') = \emptyset$
- a message set $\mathcal{M}$
- a partition of $\mathcal{X}'$ into $\{T_{x|u}(d)\}_{d \in \mathcal{M}}$ such that $\bigcup_{a \in T_{x|u}(d)} T_{z|x}(a) = \mathcal{Z}\ ,\ \forall d \in \mathcal{M}$



we can employ:

probabilistic uniform binning encoder

$$p_{x|u}(a|d) = \begin{cases} 1/N_{x|u} & , a \in T_{x|u}(d) \\ 0 & , a \notin T_{x|u}(d) \end{cases}$$

deterministic decoder

$\hat{u} = d$ if $y \in \bigcup_{a \in T_{x|u}(d)} T_{y|x}(a)$

Perfect reliability

Theorem

Let $p_{yz|x}$ be a uniform wiretap channel and $\mathcal{M}$ the message space for secret transmission over it. If:

- *$\exists \mathcal{X}' \subset \mathcal{X}$ such that $\forall a \neq a' \in \mathcal{X}'$, $T_{y|x}(a) \cap T_{y|x}(a') = \emptyset$*
- *$\exists$ a collection $\left\{T_{x|u}(d)\right\}_{d \in \mathcal{M}}$ of subsets of $\mathcal{X}'$ such that $\forall d \neq d' \in \mathcal{X}'$, $T_{x|u}(d) \cap T_{x|u}(d') = \emptyset$*
- *the random encoder satisfies $p_{x|u}(a|d) = 0$, $\forall a \notin T_{x|u}(d)$*

then there exist a decoding rule that achieves perfect reliability

Perfect reliability

Proof.

Let $T_{y|u}(d) = \cup_{a \in T_{x|u}(d)} T_{y|x}(a)$ be the subset of $\mathcal{Y}$ reachable from each $d \in \mathcal{M}$. Since the $T_{y|u}(d)$ are all disjoint, we can define the decoder

$$\hat{u} = d \quad , \quad \text{if } y \in T_{y|u}(d)$$

and we can compute the probability of correct detection as

$$\begin{aligned} \mathrm{P}\left[\hat{u} = u\right] &= \sum_{d \in \mathcal{M}} \mathrm{P}\left[\hat{u} = d | u = d\right] p_u(d) \\ &= \sum_{d \in \mathcal{M}} \mathrm{P}\left[y \in T_{y|u}(d) | u = d\right] p_u(d) = \sum_{d \in \mathcal{M}} p_u(d) = 1 \end{aligned}$$

□

Perfect secrecy

Theorem

Let $p_{yz|x}$ be a uniform wiretap channel and $\mathcal{M}$ the message space for secret transmission over it. If:

- *$\exists$ a collection $\left\{T_{x|u}(d)\right\}_{d\in\mathcal{M}}$ of subsets of $\mathcal{X}$ such that by letting $\mathcal{X}_{d\to c} = \left\{a \in \mathcal{X} \,:\, a \in T_{x|u}(d), c \in T_{z|x}(a)\right\}$ it holds $|\mathcal{X}_{d\to c}| = N \;,\; \forall c \in \mathcal{Z}, d \in \mathcal{M}$*
- *the random encoder satisfies*

$$p_{x|u}(a|d) = \begin{cases} 1/N_{x|u} & , \, a \in T_{x|u}(d) \\ 0 & , \, a \notin T_{x|u}(d) \end{cases}$$

then we have perfect secrecy of u wrt z

Perfect secrecy

Proof.

We show that u and z are independent. In fact:

$$\begin{aligned} p_{z|u}(c|d) &= \sum_{a \in \mathcal{X}} p_{z|xu}(c|a,d) p_{x|u}(a|d) \\ &= \sum_{a \in \mathcal{X}_{d \to c}} p_{z|x}(c|a) p_{x|u}(a|d) \\ &= N \frac{1}{N_{z|x}} \frac{1}{N_{x|u}} \end{aligned}$$

which is independent of the particular value d of u (and also uniform wrt $c \in \mathbb{Z}$) □

How many secret bits can be sent?

For perfect reliability to B:

$$|\mathcal{X}'| \leq \frac{|\mathcal{Y}|}{N_{y|x}}$$

For perfect secrecy with respect to E:

$$N_{x|u} \geq \frac{|\mathcal{Z}|}{N_{z|x}}$$

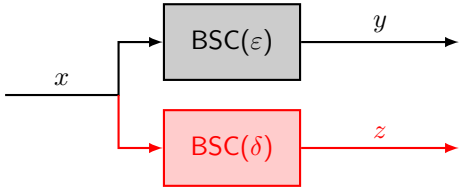
For both reliability and secrecy:

$$M = |\mathcal{M}| \leq \frac{|\mathcal{X}'|}{N_{x|u}} \leq \frac{|\mathcal{Y}|}{N_{y|x}} \frac{N_{z|x}}{|\mathcal{Z}|}$$

Secret bits in one channel use: $\log_2 M$

Secrecy capacity for the wiretap BSC

Let the channels from A to B and from A to E be memorlyess binary symmetric with error rates ε and δ, respectively



If $\left|\varepsilon - \frac{1}{2}\right| < \left|\delta - \frac{1}{2}\right|$
legitimate channel is more noisy
(e.g., $0 < \delta < \varepsilon < \frac{1}{2}$)

$$C_{\mathsf{s}} = 0$$

no secrecy is possible

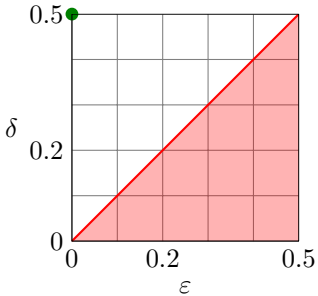
If $\left|\varepsilon - \frac{1}{2}\right| > \left|\delta - \frac{1}{2}\right|$
eavesdropper channel is more noisy
(e.g., $0 < \varepsilon < \delta < \frac{1}{2}$)

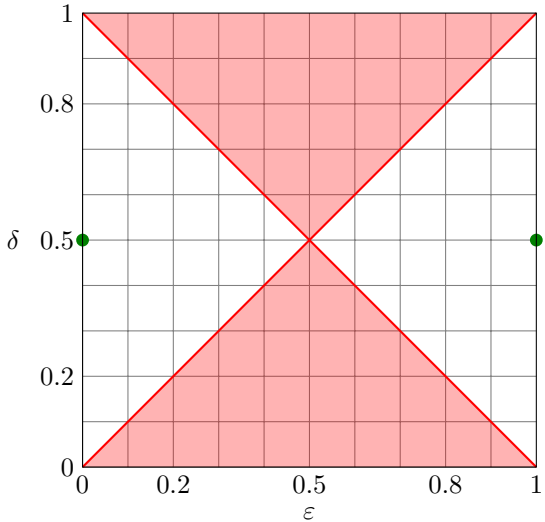
$$C_{\mathsf{s}} = C_{\mathsf{AB}} - C_{\mathsf{AE}} = h_2(\delta) - h_2(\varepsilon)$$

where $h_2(\varepsilon) = \varepsilon \log_{1/2} \varepsilon + (1-\varepsilon) \log_{1/2}(1-\varepsilon)$

Secrecy capacity for the wiretap BSC

$C_s = 0, 0.1, 0.2, 0.3, 0.4,$
$0.5, 0.6, 0.7, 0.8, 0.9, 1$
[bit/channel use]





Implement the uniform error channel

Consider a uniform error wiretap channel, where

- the input and output alphabets are the set of 7-bit words, $\mathcal{X} = \mathcal{Y} = \mathcal{Z} = \{0,1\}^7$
- the legitimate channel introduces at most 1 binary error per word,
  $T_{y|x}(a) = \{b \in \mathcal{Y} : d_{\mathsf{H}}(a,b) \leq 1\}$
  $= a \oplus \{0000000, 0000001, 0000010, 0000100, 0001000, 0010000, 0100000, 1000000\}$
- the eavesdropper channel introduces at most 3 binary error per word,
  $T_{z|x}(a) = \{c \in \mathcal{Z} : d_{\mathsf{H}}(a,c) \leq 3\}$
  $= a \oplus \{0000000, 0000001, \ldots, 1101000, 1110000\}$
- y and z are conditionally uniform and independent of each other given x

Task 1

Using a programming language of your choice, implement the wiretap channel, so that it acceps an input $x \in \mathcal{X}$ and produces the corrresponding pair of outputs (y, z)

Verify the conditional independence and uniformity of your outputs by running a sufficiently large number (at least 10^4) of channel realizations, with the same input x (e.g., $x = 1001000$) and gathering statistics

Implement the random binning encoder

Consider a uniform binning encoder, where

- the code is the $(7, 4)$ Hamming code, $\mathcal{X}' = \{0000000, 1000110, 0100101, 0010011, 0001111, 1100011, 1010101, 1001001, 0110110, 0101010, 0011100, 1110000, 1101100, 1011010, 0111001, 1111111\}$ (observe that the minimium Hamming distance in the code is 3)
- the message space is the set of 3-bit words, $\mathcal{M} = \{0, 1\}^3$
- the bin $T_{x|u}(d)$ associated to each input $d \in \mathcal{M}$ is made of 2 codewords: the one having $[0, d]$ as its 4-bit prefix, and the binary complement of the that codeword (e.g., $T_{x|u}(100) = \{0100101, 1011010\}$)
- the codeword x is chosen randomly and uniformly within the bin associated to the message u

Task 2

Using a programming language of your choice, implement the random binning encoder, so that it acceps an input $u \in \mathcal{M}$ and produces the corrresponding output $x \in \mathcal{X}'$

Verify the correctness of your implementation observing the codewords associated to different messages

Implement the random binning decoder

Consider a deterministic legitimate decoder $D : \mathcal{Y} \mapsto \mathcal{M}$, which

1. identifies the transmitted codeword by the minimum Hamming distance criterion $\hat{x} = \arg\min_{a \in \mathcal{X}'} \|a \oplus y\|_{\mathsf{H}}$
2. looks at the first bit of $\hat{x}$ and identifies the transmitted message $\hat{u}$ as either the bits 2-4 in $\hat{x}$, or their complement

Task 3

Using a programming language of your choice, implement the legitmate decoder, so that it acceps an input $y \in \mathcal{Y}$ and produces the corrresponding output $\hat{u} \in \mathcal{M}$

- Verify, by cascading encoder + decoder, that your decoder makes no errors. This is due to the property of the Hamming code being systematic.
- Verify, by cascading encoder + legitimate channel + decoder, that your decoder makes no errors. This is due to the property of the Hamming code being able to correct 1 error per word, however placed.

Verify perfect secrecy

In order to prove that our encoder achieves perfect secrecy we must show that the eavesdropper channel output z is independent of the secret message u

Task 4

- Using a programming language of your choice, implement the encoder + eavesdropper channel chain, so that it acceps an input $u \in \mathcal{M}$ and produces the corrresponding output $z \in \mathcal{Z}$.
- For each possible value of u, simulate at least $100 \cdot |\mathcal{Z}| \simeq 10^4$ realizations of the chain and gather the empirical distribution of z
- plot or tabulate the empirical conditional pmd of z given u, $\hat{p}_{z|u}(c|d)$ for all values of d
- compute the empirical joint $\hat{p}_{u,z}(d,c)$ and marginal distributions $\hat{p}_u(d), \hat{p}_z(c)$ and the mutual information

$$\hat{I}(u,z) = \sum_{d \in \mathcal{M}, c \in \mathcal{Z}} \hat{p}_{u,z}(d,c) \log_2 \frac{\hat{p}_{u,z}(d,c)}{\hat{p}_u(d)\hat{p}_z(c)}$$

Can you say that u and z are empirically independent within the statistical reliability of your simulations?

Considerations and remarks

Ponder your work and answer the following questions:

1. How many secret message bits per channel use ("transmitted word") have you obtained with your scheme?
   How many secret bits per binary digit ("transmitted bit")?
2. Is it possible to obtain 4 secret bits per channel use?
   If so, how should you change your encoder/decoder? If not, why?
3. Is it possible to obtain 2 secret bits per channel use?
   If so, how should you change your encoder/decoder? If not, why?
4. One could consider evaluating the secrecy of this mechanism by cascading the eavesdropper channel with a decoder and measuring the resulting error rates. What do you expect Eve's error would be? Why resort to (more complicated) evaluating the mutual information?

Simulate transmission over a binary symmetric channel

Consider a wiretap binary symmetric channel, with independent errors on the two branches, and error rates ε and δ for the legitimate and eavsesdropper channel, respectively.

Task 5

Using a programming language of your choice, implement the wiretap BSC, so that

- it can be simulated with arbitrary values of ε, δ,
- it can be connected in between the random binning encoder developed in Task 2, and the decoder developed in Task 3

Verify the correctness of your implementation by transmitting a long binary sequence and checking the number of bit errors in each output.

Connect the wiretap channel to the random binning encoder and the legitimate decoder, and simulate several transmissions.

Observe that perfect reliability or secrecy are no longer provided, as there may be more than one error in a single codeword over the legitimate channel, and the error pattern distribution in the eavesdropper channel is no longer uniform.

Evaluate the system security over the wiretap BSC

Task 6

Choose several values of ε and δ, and for each (ε, δ) pair

- repeat the simulations in Tasks 3-4 with the wiretap BSC
- evaluate the resulting reliability in terms of Bob's error rate on the secret message $\mathrm{P}\left[u \neq \hat{u}\right]$
- evaluate the resulting the secrecy in terms of leaked information to Eve on the secret message $I(u; z)$
- compute an upper bound to the mechanism security in terms of distinguishability from the ideal counterpart

What you need to turn in

Each team must turn in, through the Moodle assignment submission procedure:

1. the source code for your implementation (either as a single file, many separate files, or a compressed folder)
2. a short report (to be submitted as a separate file from the source code file / compressed folder) in a graphics format (PDF, DJVU or PostScript are ok; Word, TEX or LATEX source are not), including:
   - 2.1 a brief description of your implementations for Tasks 1-6, explaining your choices;
   - 2.2 your answers to the questions in Considerations and remarks
   - 2.3 the evaluated security metrics for your system:
     - 2.3.1 a plot of the conditional pmd $p_{z|x}(\cdot|1001000)$ from Task 1;
     - 2.3.2 the plots of the conditional pmds $p_{z|u}(\cdot|d)$ for all values of d from Task 4;
     - 2.3.3 the estimates of $H(u)$, and $I(u;z)$ from Task 4
     - 2.3.4 a plot of the error decoding probability $\mathrm{P}\left[\hat{u} \neq u\right]$ as a function of ε for the BSC from Task 6
     - 2.3.5 a plot of the mutual information $I(u;z)$ as a function of δ for the BSC from Task 6
     - 2.3.6 a contour plot of the security of this mechanism as a function of ε and δ for the BSC from Task 6

Appendix: Hamming codes

Hamming codes are linear binary codes with the following properties:

1. There exist Hamming codes with dimension k and length n (i.e., with 2^k codewords of n bits) if and only if $n + 1 = 2^{n-k}$
2. A generating matrix for a Hamming code of length n is given by $\boldsymbol{G} = \left[\frac{\boldsymbol{I}_k}{\boldsymbol{A}}\right]$ where $\boldsymbol{I}_k$ is the $k \times k$ identitiy matrix and the k columns of $\boldsymbol{A}$ are all the words of $n - k$ bits with Hamming weight ≥ 2
3. The minimum Hamming distance between any two codewords is $d_{\text{min}} = 3$
4. Every binary word in $\{0, 1\}^n$ is either a codeword or at distance 1 from a codeword

As a consequence of the above properties 3-4, when used in forward error correction over a binary channel:

- a Hamming code can exactly correct 1 error in each codeword, however placed;
- a Hamming code has the highest rate (i.e., maximum number of codewords, maximum information carried) among all the single-error correcting codes of the same length n.